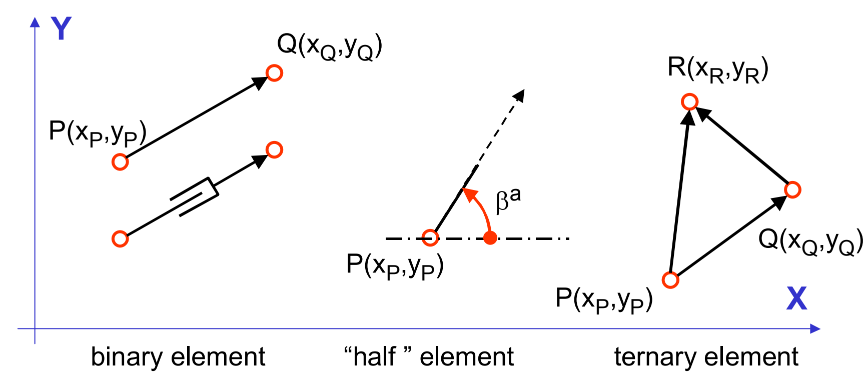
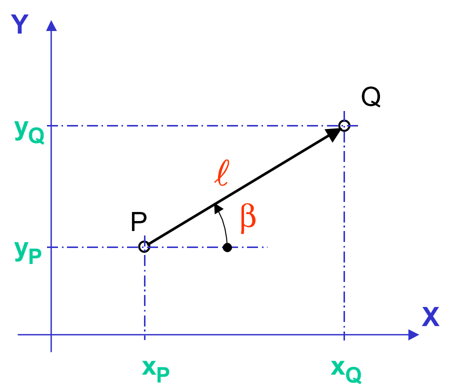


$$\underline{x}^T = |x_P\ y_P\ x_Q\ y_Q| \qquad \underline{x}^T = |x_P\ y_P\ \beta^a| \qquad \underline{x}^T = |x_P\ y_P\ x_Q\ y_Q\ x_R\ y_R|$$

Fig.3.1.1 Planar kinematic elements and generalised co-ordinates



$$\ell = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$$

$$\cos\beta = \frac{x_Q - x_P}{\ell}$$

$$\sin\beta = \frac{y_Q - y_P}{\ell}$$

*Fig. 3.1.2 Length ℓ and angle β of a binary element dependent on the co-ordinates **x***

3 Elementary kinematics and differential geometry

3.1 Introduction

Kinematic analysis deals with the determination of the motion of a given mechanism. Given means that both the type and the kinematic dimensions are known. In this chapter it will also be assumed that the mechanism is in a given position. In that case the positions of all elements are known. The attention can thus be focussed on the kinematic properties of a single element in a given position.

The way elements are defined needs an explanation. Kinematic problems usually play a role in the very early design stage of a mechanism (see fig. 1.3.1). In kinematics it is usually not important how the precise shape of an element is. In co-planar mechanisms the elements are conceptually just moving planes with undefined edges. Typically a slider has a short but undefined length and the slot length is long enough to guide the slider. Typically the planes of all elements can move without notice of collision.
The position and the dimensions of an element can be described then with a limited number of points or other geometrical quantities. Preparing for the Finite Element approach (chapter 5) the describing quantities will be called ***generalised co-ordinates***, to be contained in a vector $\underline{x}$. The word generalised means here that quantities of different physical meaning can be used. Examples of valid planar elements have been depicted in fig. 3.1.1.

In accordance with the basic rules of FEM it is assumed that

- Co-ordinates are defined in the global co-ordinate system (the frame).
- Co-ordinates can be varied independently of each other (can be considered mathematically as a set of independent variables).
- Their number must minimally be 3 (in spatial kinematics 6). More co-ordinates than the minimum number is allowed, but then one or more relations must exist between the co-ordinates. A fixed length of a binary element is such a relation.

The theory concerning relations between the co-ordinates and the consequences for the motion of the whole mechanism will be given attention further in chapter 5. In this chapter it will be assumed that the co-ordinates of an element are just given quantities. As motion analysis includes derivatives like velocity and acceleration, it must be assumed that derivatives of the generalised co-ordinates are, when required, also given quantities.
In this book it will be preferred to specify these derivatives with respect to the degree(s) of freedom. This approach is indicated in literature as: ***differential geometry***. This is the base for time derivatives (as was shown in chapter 1.4).
A typical way of thinking is depicted in figure 3.1.2. The element is represented by two points P and Q (binary element). The co-ordinate values of these two points allow calculating the length ℓ or the angle β. When a third point R is attached to this plane (to be defined for instance with local co-ordinates u and v), the path (x_R, y_R) of this point can be expressed like in figure 3.1.3.

Summary:

- *A kinematic planar element is a plane with undefined shape. Its position is defined by a set of (generalised) co-ordinates in the global co-ordinate system. Its motion is determined by the motion of co-ordinates.*
- *In this chapter the co-ordinates of each element and their derivatives will be treated as given quantities, with which kinematic properties of that element will be determined.*

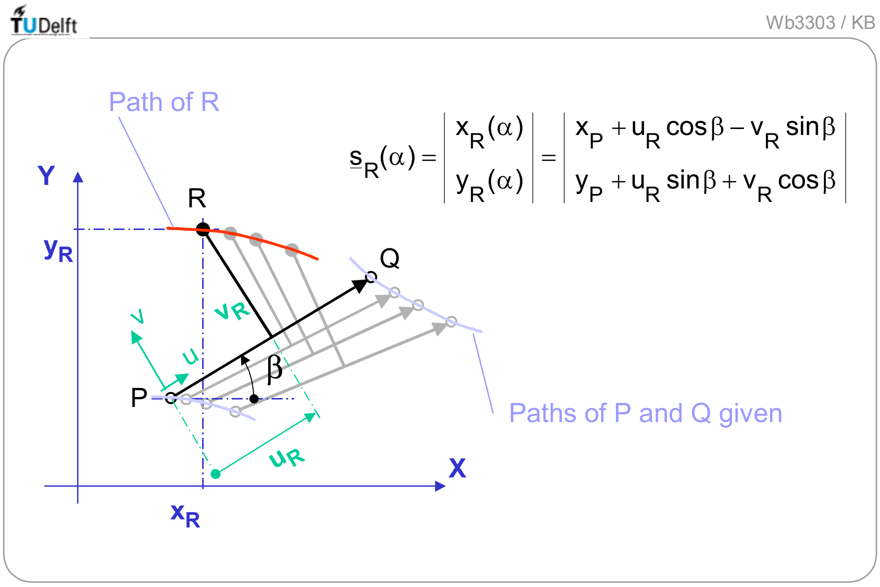


Fig. 3.1.3 *Path of a point R attached to a binary element, using local co-ordinates u, v*

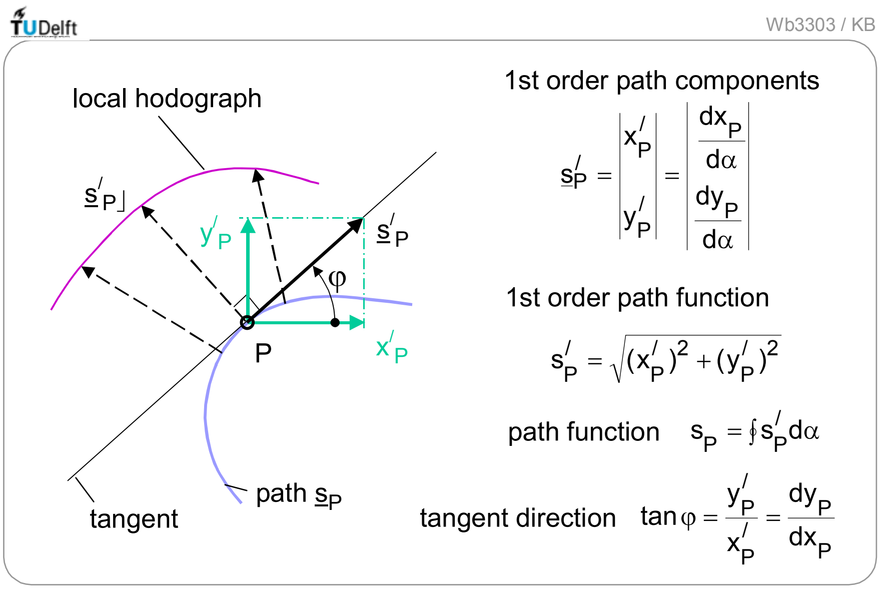


Fig. 3.2.1 *First order motion of a point P*

3.2 Motion quantities, definitions

In the previous chapter it was noticed that kinematic analysis often requires study of derivative motion quantities. The derivatives of the generalised co-ordinates itself, like the first or second order path components of a point P, can be of interest, see figs. 3.2.1 and 3.2.2. There are many other quantities that can play a role in expressing motion and that need to be determined then.

- Motion quantities that are a ***property of a point P*** are the following.

Tangential direction φ (motion direction), see fig. 3.2.1

First order path function $s^{/}$, the size of the first order motion vector, see fig. 3.2.1. Sometimes this quantity is called geometric velocity. Note that the velocity of point P is

$$\dot{s}_P = \frac{ds_P}{dt} = \frac{ds_P}{d\alpha} \cdot \frac{d\alpha}{dt} = s^{/}_P \cdot \dot{\alpha} = \dot{\alpha} \cdot \sqrt{(x^{/}_P)^2 + (y^{/}_P)^2} \qquad (3.1)$$

The path function s is the path length as covered by the point after being displaced. This quantity can be defined with the first order path function (the integral of $s^{/}$, along the path). For a closed curve it is the circular integral.

Radius of curvature ρ. Its inverse is the curvature, which is by definition $d\varphi/ds$. In literature it can be found that

$$\rho_P = \frac{(s^{/}_P)^3}{x^{/}_P y^{//}_P - x^{//}_P y^{/}_P} \qquad (3.2)$$

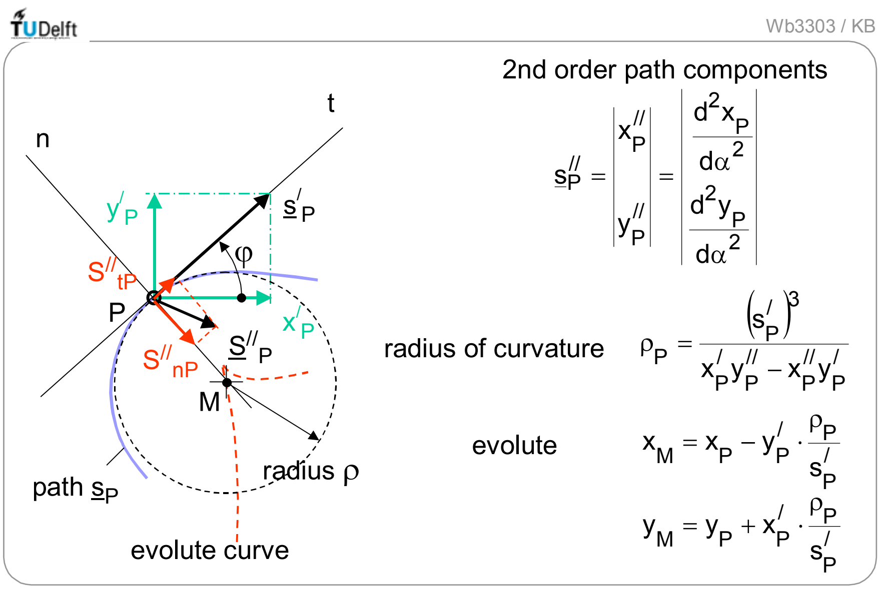


Fig. 3.2.2 Second order motion of a point P

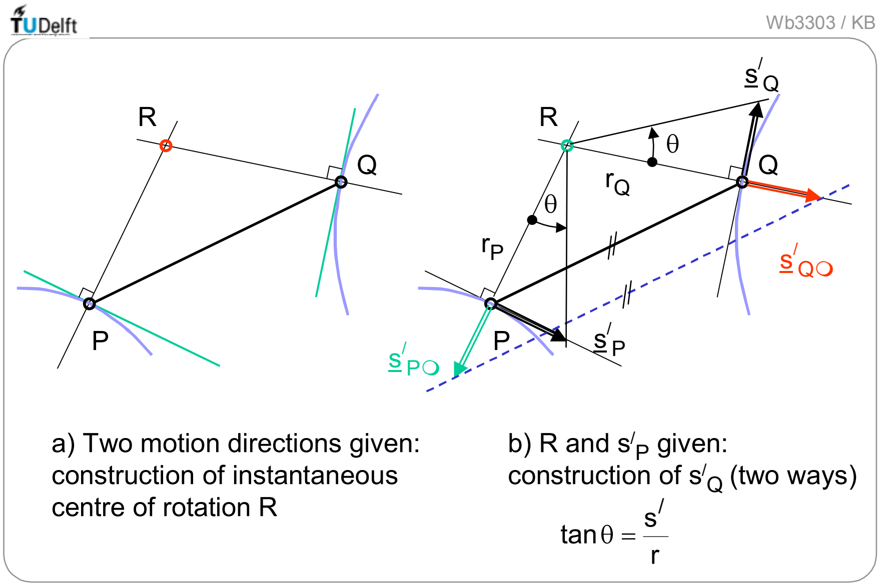


Fig. 3.2.3 Instantaneous centre of rotation R

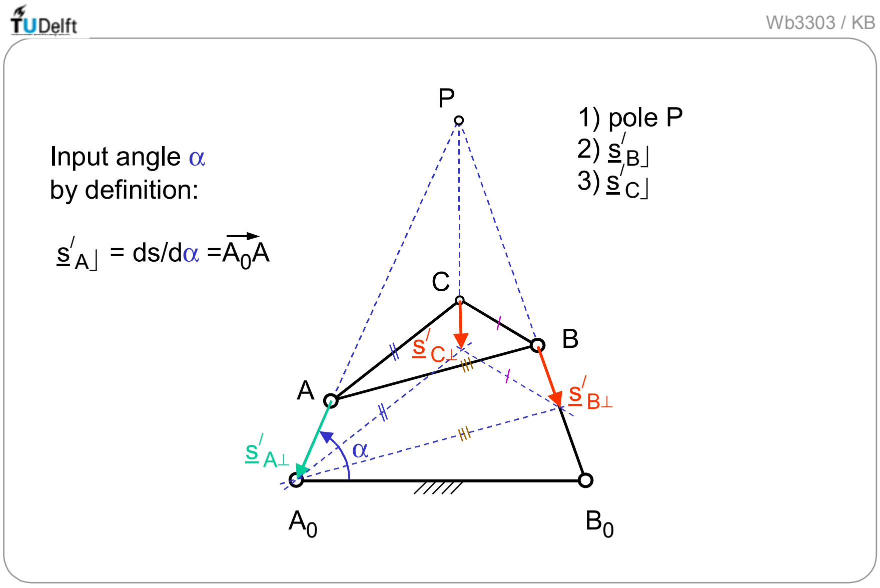


Fig. 3.2.4 Construction of first order motion of a coupler point (acc. Mehmke)

The centre point of the *osculation circle* (x_M, y_M), the circle that has the radius of curvature and which fits therefore best to the curve in P, can be expressed with

$$x_M = x_P - y_P^{/} \cdot \frac{\rho_P}{s_P^{/}} \quad ; \quad y_M = y_P + x_P^{/} \cdot \frac{\rho_P}{s_P^{/}} \tag{3.3}$$

The locus of points (x_M, y_M) is called the *evolute curve*. It shows a characteristic cusp where the curvature has a minimum or maximum, and it has an asymptote where the curvature changes its sign.

- Motion quantities that are a ***property of the element*** are the following.

The *(instantaneous) centre of rotation (first order pole)*. When two points of one element have a known motion direction, then this pole is determined by the intersection of the normal lines (see fig. 3.2.3 point R). The centre of rotation can be used advantageously to find the first order motion of other points of the plane. In fig. 3.2.3 right part it is assumed that $\underline{s}^{/}_P$ and pole R are given. Now $\underline{s}^{/}_Q$ can be constructed easily, observing that from R all first order vectors will be seen at the same angle θ. A comfortable way to carry out the graphical construction applies the vector $\underline{s}^{/}_P$ rotated 90˚. Now a line through the tip of this vector, drawn parallel to PQ, and intersecting the line RQ gives the (rotated) vector $\underline{s}^{/}_Q$.
In figure 3.2.4 this graphical construction is used to find the first order vectors of the two points B and C of the coupler plane of a four-bar linkage. Note that, due to the parallel lines:

> ***The polygon of the coupler points and the polygon of the end-points of their (rotated) first order vectors are conform.***

First order angular motion β/. From fig. 3.2.3 it follows that $\beta^{/} = \tan\theta = s^{/}/r$, where r is the distance to the pole. Another way to express $\beta^{/}$ uses the first order motion components of two points, P and Q in fig. 3.2.5, perpendicular to PQ. For this purpose a new global co-ordinate system (x*, y*) has been chosen such that PQ is instantaneous along the x*-axis.

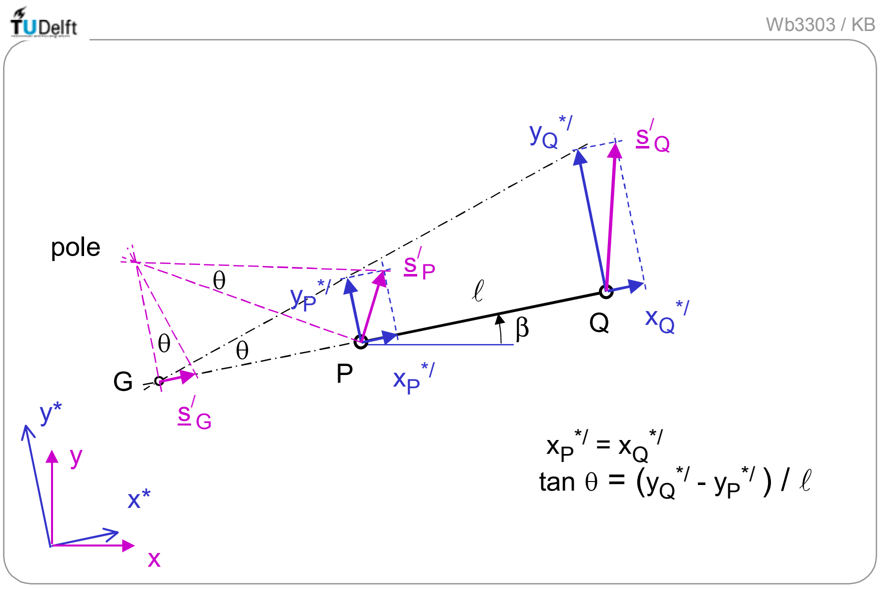


Fig. 3.2.5 First order motion along a bar is the same at any point

TUDelft Wb3303 / KB

Subscr. a: absolute motion
f: frame motion
r: relative motion

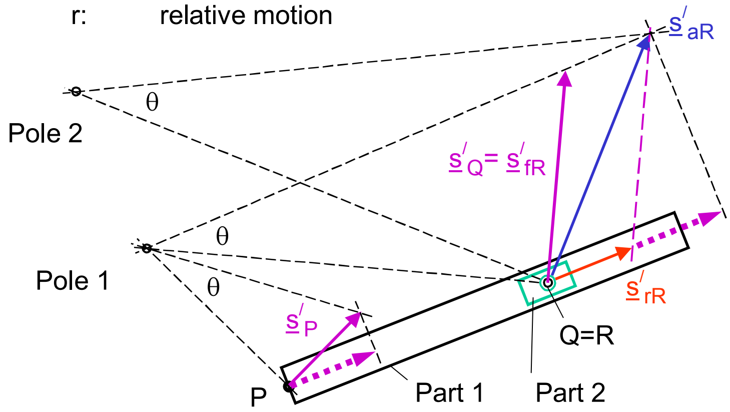


Fig. 3.2.6 Relative sliding motion components, first order (P and Q at part 1, R at part 2, Q and R coincide)

TUDelft Wb3303 / KB

$$x_Q - x_P = \ell \cdot \cos\beta$$
$$y_Q - y_P = \ell \cdot \sin\beta$$

Differentiate w.r.t. to α

$$x'_Q - x'_P = \ell' \cdot \cos\beta - \ell\beta' \cdot \sin\beta$$
$$y'_Q - y'_P = \ell' \cdot \sin\beta + \ell\beta' \cdot \cos\beta$$

Write ℓ' and β' explicitly

$$\ell' = +(x'_Q - x'_P)\cos\beta + (y'_Q - y'_P)\sin\beta$$
$$\beta' = -(x'_Q - x'_P)\frac{\sin\beta}{\ell} + (y'_Q - y'_P)\frac{\cos\beta}{\ell}$$

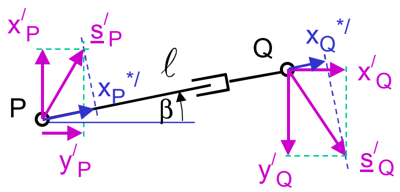


Fig. 3.3.1 Analytical derivation of relative motion (1st order)

First order motion of points at a line of the element. In fig. 3.2.5 it can be recognised easily that

> ***All points on a line of an element have the same first order motion in the direction of that line.***

- Motion quantities of ***relative motion*** are the following.

First order elongation $\ell^{/}$ .Of course a bar with fixed length will have $\ell^{/} = 0$. A slider pair can however also be defined by just two points P and R, see fig. 3.2.6, of which the elongation can be nonzero. Now there are two elements (parts) having the same rotation. Part 2 (the slider with point R) is assumed to move relatively to part 1 (the slot with point P). The motion of point R can be decomposed then into a *relative motion* (along PR) and a so-called *frame motion*. Frame motion occurs when point R would be a point of part 1. Perhaps this idea can better be understood introducing a third point Q that is connected at part 1 at the place where point R is in the situation to be considered. By definition the frame motion of point R is then the motion of point Q. The vector sum of frame motion and relative motion of point R is the absolute motion of point R.
Note that the first order motion of points P and Q along the bar are equal.
Note that the slider part has the same rotation as the slot part. Consequently the values of $\beta^{/}$ (and thus the angles θ) are the same. The first order pole of part 2 can thus be found easily when the pole of part 1 is known.
Relative motion will be considered in more detail in the next chapter.

3.3 Motion quantities, analytical approach

Motion quantities can often very well be determined with graphic constructions. Nevertheless it can be helpful to have the formulas available. These formulas will be derived in this chapter. The approach will be that a bar with fixed length and a bar with variable length (having relative motion) will be treated simultaneously. This can be done understanding that fixed length is a special situation of the general case with variable length. The idea of a bar with variable length is a simplification of the relative motion idea as represented by a slider pair, which consists physically of two parts.

The bar with variable length will be represented here by the two points P and Q. The (given) co-ordinates of these points determine the position of this binary element. The derivatives of these co-ordinates can be understood as independent variables also, and they determine the derivative motion quantities of the element (including relative motion).

3.3.1 Binary element: length ℓ and angle β and their derivatives.

Starting with the relations, see also fig. 3.3.1

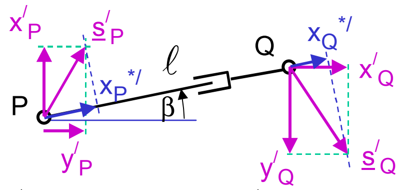
$$\begin{aligned} x_Q - x_P &= \ell \cdot \cos\beta \\ y_Q - y_P &= \ell \cdot \sin\beta \end{aligned} \tag{3.4}$$

it can be found easily that

$$\begin{aligned} \ell &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} \\ \tan\beta &= \frac{y_Q - y_P}{x_Q - x_P} \end{aligned} \tag{3.5}$$

Wb3303 / KB

$$x'_Q - x'_P = \ell' \cdot \cos\beta - \ell\beta' \cdot \sin\beta$$
$$y'_Q - y'_P = \ell' \cdot \sin\beta + \ell\beta' \cdot \cos\beta$$



2nd derivation

$$x''_Q - x''_P = \ell'' \cdot \cos\beta - 2\ell'\beta' \cdot \sin\beta - \ell\left\{\beta''\sin\beta + (\beta')^2\cos\beta\right\}$$
$$y''_Q - y''_P = \ell'' \cdot \sin\beta + 2\ell'\beta' \cdot \cos\beta + \ell\left\{\beta''\cos\beta - (\beta')^2\sin\beta\right\}$$

Write ℓ'' and β'' explicitly

$$\ell'' = +(x''_Q - x''_P)\cos\beta + (y''_Q - y''_P)\sin\beta + \ell(\beta')^2$$

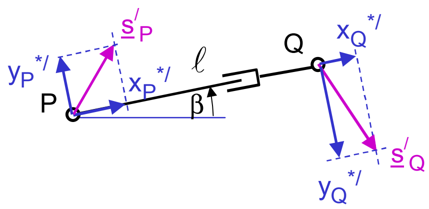
$$\beta'' = -(x''_Q - x''_P)\frac{\sin\beta}{\ell} + (y''_Q - y''_P)\frac{\cos\beta}{\ell} - \frac{2\ell'\beta'}{\ell}$$

Fig. 3.3.2 Analytical derivation of relative motion (2nd order)

Wb3303 / KB

New co-ordinate system:

$$x^{*\prime} = \ x'\cos\beta + y'\sin\beta$$
$$y^{*\prime} = -x'\sin\beta + y'\cos\beta$$



Substitution:

$$\ell' = x^{*\prime}_Q - x^{*\prime}_P \qquad \ell'' = x^{*\prime\prime}_Q - x^{*\prime\prime}_P + \ell(\beta')^2$$

$$\beta' = \frac{y^{*\prime}_Q - y^{*\prime}_P}{\ell} \qquad \beta'' = \frac{y^{*\prime\prime}_Q - y^{*\prime\prime}_P}{\ell} - \frac{2\ell'\beta'}{\ell}$$

Fig. 3.3.3 Transformation to new co-ordinate system

Now eq.(3.4) can be differentiated with respect to the degree of freedom of the mechanism:

$$\begin{aligned} x_Q^{/} - x_P^{/} &= \ell^{/} \cdot \cos\beta - \ell\beta^{/} \cdot \sin\beta \\ y_Q^{/} - y_P^{/} &= \ell^{/} \cdot \sin\beta + \ell\beta^{/} \cdot \cos\beta \end{aligned} \tag{3.6}$$

in which $\beta^{/}$ and $\ell^{/}$ are considered as the two unknowns of the two equations (3.6). For a given position of the element the values of P and β are calculable according (3.5), so (3.6) contains two linear equations. To write $\ell^{/}$ explicitly the first equation, multiplied with cosβ, can be added to the second, multiplied with sinβ. To write $\beta^{/}$ explicitly the second equation, multiplied with cosβ, can be subtracted from the first one, multiplied with sinβ. The result becomes (see also fig. 3.3.1):

$$\begin{aligned} \ell^{/} &= +(x_Q^{/} - x_P^{/})\cos\beta + (y_Q^{/} - y_P^{/})\sin\beta \\ \beta^{/} &= -(x_Q^{/} - x_P^{/})\frac{\sin\beta}{\ell} + (y_Q^{/} - y_P^{/})\frac{\cos\beta}{\ell} \end{aligned} \tag{3.7}$$

Second derivation of eq.(3.4) yields

$$\begin{aligned} x_Q^{//} - x_P^{//} &= \ell^{//} \cdot \cos\beta - 2\ell^{/}\beta^{/} \cdot \sin\beta - \ell\left\{\beta^{//}\sin\beta + (\beta^{/})^2\cos\beta\right\} \\ y_Q^{//} - y_P^{//} &= \ell^{//} \cdot \sin\beta + 2\ell^{/}\beta^{/} \cdot \cos\beta + \ell\left\{\beta^{//}\cos\beta - (\beta^{/})^2\sin\beta\right\} \end{aligned} \tag{3.8}$$

in which $\beta^{//}$ and $\ell^{//}$ are considered as the two unknowns. For given position and first order motion (3.8) contains two linear equations from which $\beta^{//}$ and $\ell^{//}$ can be solved. The result becomes (see also fig. 3.3.2):

$$\begin{aligned} \ell^{//} &= +(x_Q^{//} - x_P^{//})\cos\beta + (y_Q^{//} - y_P^{//})\sin\beta + \ell(\beta^{/})^2 \\ \beta^{//} &= -(x_Q^{//} - x_P^{//})\frac{\sin\beta}{\ell} + (y_Q^{//} - y_P^{//})\frac{\cos\beta}{\ell} - \frac{2\ell^{/}\beta^{/}}{\ell} \end{aligned} \tag{3.9}$$

The formulas derived above become much simpler when the (global) co-ordinate system is taken along the bar. This new co-ordinate system will be noted as (x^*, y^*). Recognising that the transformation between both co-ordinate systems can be expressed by

$$\begin{aligned} x^{*/} &= +x^{/}\cos\beta + y^{/}\sin\beta \\ y^{*/} &= -x^{/}\sin\beta + y^{/}\cos\beta \end{aligned} \quad \text{and} \quad \begin{aligned} x^{*//} &= +x^{//}\cos\beta + y^{//}\sin\beta \\ y^{*//} &= -x^{//}\sin\beta + y^{//}\cos\beta \end{aligned} \tag{3.10}$$

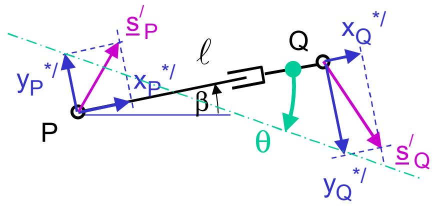
it can be detected easily that eq.(3.7) can be written as

$$\begin{aligned} \ell^{/} &= x_Q^{*/} - x_P^{*/} \\ \beta^{/} &= \frac{y_Q^{*/} - y_P^{*/}}{\ell} \end{aligned} \tag{3.11}$$

and eq.(3.9) as

$$\begin{aligned} \ell^{//} &= x_Q^{*//} - x_P^{*//} + \ell(\beta^{/})^2 \\ \beta^{//} &= \frac{y_Q^{*//} - y_P^{*//}}{\ell} - \frac{2\ell^{/}\beta^{/}}{\ell} \end{aligned} \tag{3.12}$$

Wb3303 / KB



$$\begin{aligned} x_Q^{*\prime} &= x_P^{*\prime} + 0 + \ell^{\prime} \\ y_Q^{*\prime} &= y_P^{*\prime} + \ell\beta^{\prime} + 0 \end{aligned}$$

Abs frame relative

$$\beta^{\prime} = \frac{y_Q^{*\prime} - y_P^{*\prime}}{\ell} = \tan\theta$$

$$\begin{aligned} x_Q^{*\prime\prime} &= x_P^{*\prime\prime} - \ell(\beta^{\prime})^2 + \ell^{\prime\prime} + 0 \\ y_Q^{*\prime\prime} &= y_P^{*\prime\prime} + \ell\beta^{\prime\prime} + 0 + 2\beta^{\prime}\ell^{\prime} \end{aligned}$$

$$\ell(\beta^{\prime})^2 = \frac{(y_Q^{*\prime} - y_P^{*\prime})^2}{\ell}$$

Abs frame relative Coriolis

Fig. 3.3.4 Decomposition of relative motion components

For a bar with fixed length it must be true that $\ell^{/} = 0$. It can be concluded from (3.11) that $x_P^{*/} = x_Q^{*/}$. This result (the first motion along the bar is the same for all points) was already found in the previous chapter. Another result is that the term with $\ell^{/}\,\beta^{/}$ vanishes, but the term $\ell(\beta^{/})^2$ does not. Note that $\ell^{//}$ must also be zero and that $x_P^{*//}$ *does not equal* $x_Q^{*//}$ .Here is a difference with first order motion along the bar!

For a bar with variable length the term with $\ell^{/}\,\beta^{/}$ is characteristic for relative motion (Coriolis term). To understand now better the decomposition into frame motion and relative motion, eqs. (311) and (3.12) can be rewritten as (see also fig. 3.3.4)

$$\begin{aligned} x_Q^{*/} &= x_P^{*/} + 0 + \ell^{/} \\ y_Q^{*/} &= y_P^{*/} + \ell\beta^{/} + 0 \end{aligned} \tag{3.13}$$

$$\begin{aligned} x_Q^{*//} &= x_P^{*//} - \ell(\beta^{/})^2 + \ell^{//} + 0 \\ y_Q^{*//} &= y_P^{*//} + \ell\beta^{//} + 0 + 2\beta^{/}\ell^{/} \end{aligned} \tag{3.14}$$

So far it was succeeded to derive some important motion quantities like the angular derivatives. The first angular derivative $\beta^{/}$ can for instance simply be determined from the first order motion of the points P and Q (components perpendicular to the bar), even in case that the bar length is not constant. Second order motion introduces some other terms, which will be considered further in the next chapter (3.4)

3.3.2 First and second order pole.

The co-ordinates of a coupler point can be expressed, see also fig. 3.1.3, as

$$\begin{aligned} x_R &= x_P + u_R\cos\beta - v_R\sin\beta \\ y_R &= y_P + u_R\sin\beta + v_R\cos\beta \end{aligned} \tag{3.15}$$

Differentiation yields:

$$\begin{aligned} x_R^{/} &= x_P^{/} - u_R\beta^{/}\sin\beta - v_R\beta^{/}\cos\beta \\ y_R^{/} &= y_P^{/} + u_R\beta^{/}\cos\beta - v_R\beta^{/}\sin\beta \end{aligned} \tag{3.16}$$

To find the *first order pole* R it must be assumed that $x^{/}_R = y^{/}_R = 0$. Now, for given position and given first order motion of the element, (3.16) contains two linear equations in the unknown's u_R and v_R, which can be solved as:

$$\begin{aligned} u_R &= u_{pole} = \frac{x_P^{/}\sin\beta - y_P^{/}\cos\beta}{\beta^{/}} \\ v_R &= v_{pole} = \frac{x_P^{/}\cos\beta + y_P^{/}\sin\beta}{\beta^{/}} \end{aligned} \tag{3.17}$$

This determines the place of the first order pole in the coupler plane.

Substitution of (3.17) in (3.15) leads to the pole position in the global co-ordinate system:

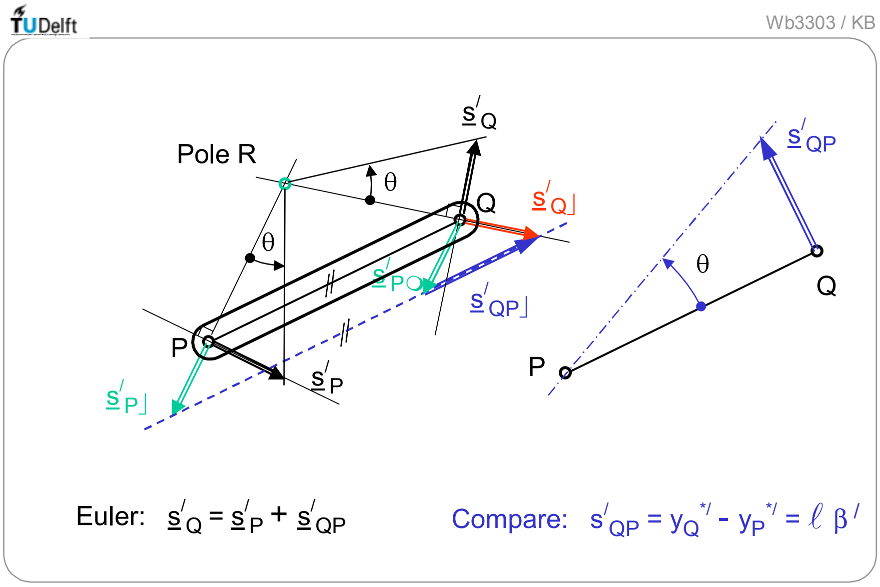


Fig. 3.4.1 Decomposition of first order motion of a binary element (acc. Euler)

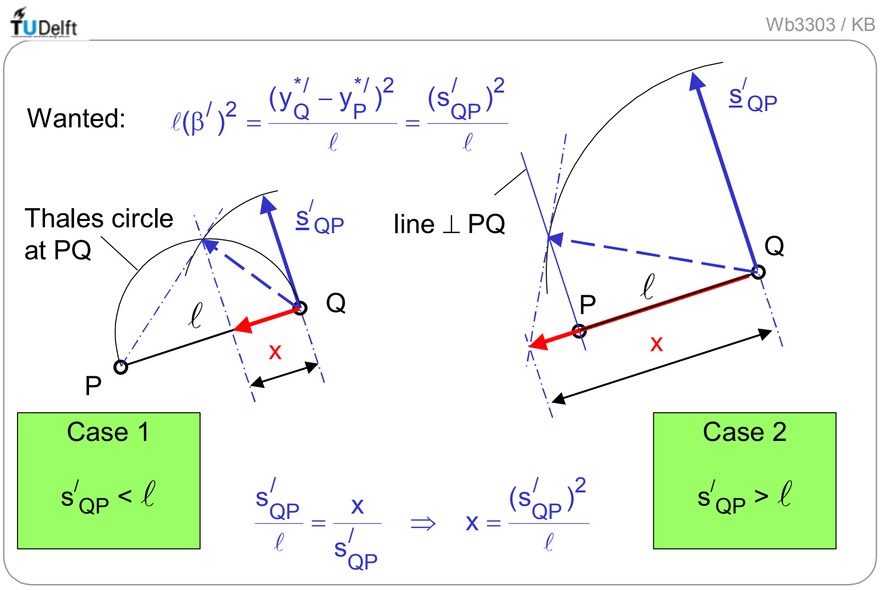


Fig. 3.4.2 Normal component of second order motion of a binary link, graphical construction

$$x_R = x_{pole1} = x_P - \frac{y_P^{/}}{\beta^{/}}$$
$$y_R = y_{pole1} = y_P + \frac{x_P^{/}}{\beta^{/}} \qquad (3.18)$$

The locus of pole positions (x_R, y_R) in the fixed plane is called the ***fixed polode*** (or fixed centrode). The locus in the moving plane (u_R, v_R) is called the ***moving polode*** (or moving centrode).

Remark. It is very well possible that $\beta^{/} = 0$ in a certain position. Any oscillating motion has such positions for the extreme angular positions of the plane. In that case the first order pole lies in infinity. The equations in (3.17) and (3.18) should be used with care in a computer program, because a division by zero is possible.

The second order motion components of a coupler point can be obtained by differentiation of (3.16):

$$x_R^{//} = x_P^{//} + u_R(\beta, \beta^{/}, \beta^{//}) + v_R(\beta, \beta^{/}, \beta^{//})$$
$$y_R^{//} = y_P^{//} + u_R(\beta, \beta^{/}, \beta^{//}) + v_R(\beta, \beta^{/}, \beta^{//})$$

The second order pole, the point with by definition $x^{//}_R = y^{//}_R = 0$, can be solved from this system of two linear equations and the two unknowns u_R and v_R (it will be clear that the motion of this element up to order two must be given). Substitution of the result in (3.15) finally gives the position of the second order pole in the fixed co-ordinate system:

$$x_R = x_{pole2} = x_P + \frac{x_P^{//}(\beta^{/})^2 - y_P^{//}\beta^{//}}{(\beta^{/})^4 + (\beta^{//})^2}$$
$$y_R = y_{pole2} = y_P + \frac{y_P^{//}(\beta^{/})^2 - x_P^{//}\beta^{//}}{(\beta^{/})^4 + (\beta^{//})^2} \qquad (3.19)$$

Note that a zero denominator occurs only when both $\beta^{/}$ and $\beta^{//}$ are zero. This will happen only in very special situations.

Now we look again at eq (3.13) and suppose that point P of the moving plane coincides with the first order pole ($x^{*/}_P$ and $y^{*/}_P$ are zero). The first order motion components of any point Q at a fixed distance ℓ to P, in a co-ordinate system with x^* along PQ, are then:
$x^{*/}_Q = 0$ and $y^{*/}_Q = \ell\beta^{/}$. It can be concluded that

> ***The first order motion field of a kinematic plane is a radial, linear field with the vectors perpendicular to the radii.***

A comparable conclusion can be drawn from eq.(3.14). Suppose point P coincides with the second order pole ($x^{*//}_P$ and $y^{*//}_P$ are zero). The second order motion components of any point Q at fixed distance ℓ to P, in a co-ordinate system with x^* along PQ, are then:
$x^{*//}_Q = \ell(\beta^{/})^2$ and $y^{*//}_Q = \ell\beta^{//}$. It can be concluded that

> ***The second order motion field of a kinematic plane is a radial, linear field with all vectors at the same angle ψ to the radii.***

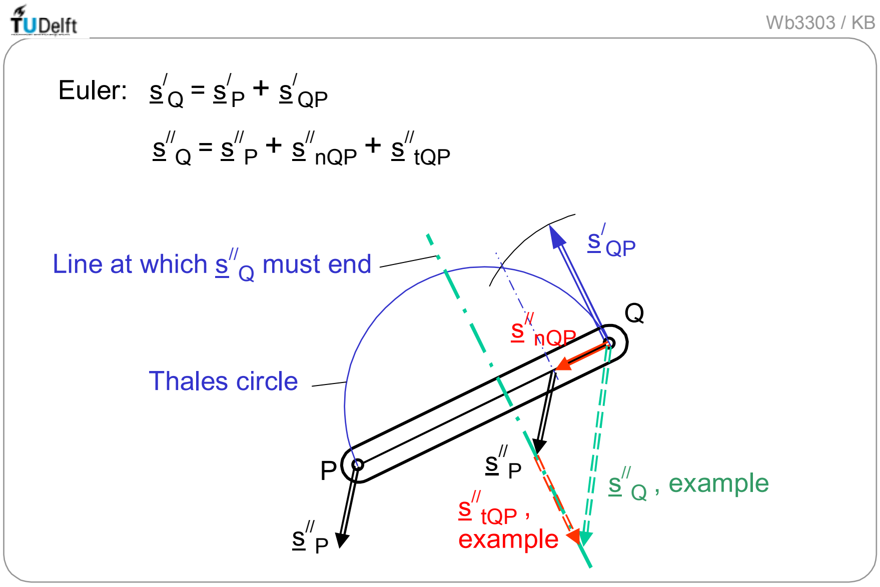


Fig. 3.4.3 Graphical basic procedure for construction of second order motion (acc. Euler)

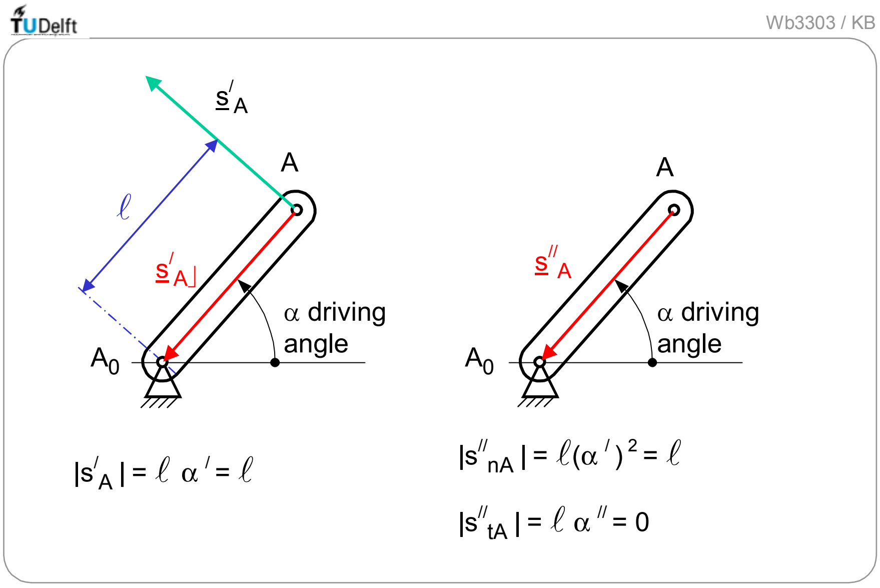


Fig. 3.4.4 Motion components of a crank point

The angle ψ is acute and can be expressed by

$$\tan\psi = \frac{\beta^{//}}{(\beta^{/})^2} \tag{3.20}$$

The idea of the second order pole is depicted in fig. 3.4.6 (right part).

3.4 Motion quantities, graphical approach

Graphical constructions up to first order motion have been treated in chapter 3.2. Now the graphical constructions of second order motion will be regarded. Maybe this looks overdone, since all relevant second order motion quantities of a moving element can be determined analytically (chapter 3.3). This assumes however that the motions of the generalised co-ordinates of the element up to the second order are given quantities. In literature the graphical approach has extensively be treated, not only to analyse the kinematic behaviour of a single element, but also to find the motion of the whole mechanism. The basic principles of the graphical approach will be considered therefore in this chapter.

3.4.1 Decomposition according to Euler

Euler considered the motion of a bar PQ with fixed length. He assumed that the (first order) motion of point P and the rotation of the bar are given. The motion of any further point of the moving plane Q can be expressed then as a superposition of the motion of P and the rotation around P. In vector notation, see fig. 3.4.1:

$$\underline{s}^{/}_{Q} = \underline{s}^{/}_{P} + \underline{s}^{/}_{QP} \tag{3.21}$$

From eq. (3.13) it can be detected that $s^{/}_{QP} = \ell_{PQ}\beta^{/} = y^{*/}_{Q} - y^{*/}_{P}$. In the graphical approach the vector $\underline{s}^{/}_{QP}$ can be found by drawing a closed vector triangle. This can also be carried out with the vectors rotated over 90 degrees. The second order proceeds then with

$$\underline{s}^{//}_{Q} = \underline{s}^{//}_{P} + \underline{s}^{//}_{QP} = \underline{s}^{//}_{P} + \underline{s}^{//}_{nQP} + \underline{s}^{//}_{tQP} \tag{3.22}$$

From eq. (3.14) it can be detected that $s^{//}_{nQP} = \ell_{PQ}(\beta^{/})^2 = (s^{/}_{QP})^2/\ell_{PQ}$ and that $s^{//}_{tQP} = \ell_{PQ}\beta^{//}$. Note that the first term *depends only on the first order motion*. The graphical construction is depicted in fig. 3.4.2.

As long as the second order rotation $\beta^{//}$ of the element is unknown, the vector sum of second order vectors can not be completed. However, the locus at which $\underline{s}^{//}_{QP}$ must end, can be drawn as a line perpendicular to PQ, see fig. 3.4.3. This result will be used later in several examples.

Example 1, the mechanism consists of a rotating crank, see fig. 3.4.4. The motion of point A_0 is given (all motion components up to order two are zero and point A_0 is both the first and the second order pole). Because the angle α of the crank is the driving angle, $\alpha^{/} = 1$ and $\alpha^{//} = 0$. It follows then that both $\underline{s}^{/}_{A}$ and $\underline{s}^{//}_{A}$ have crank length A_0A.

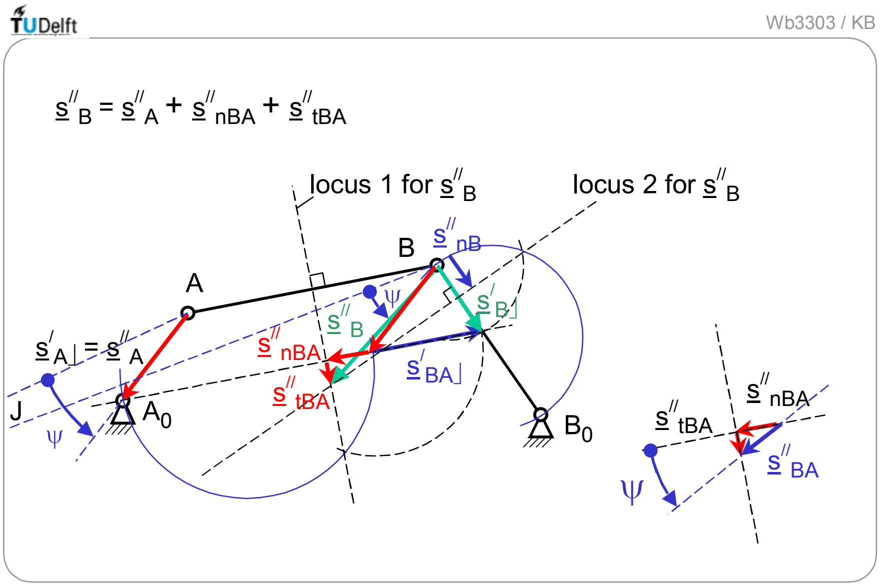


Fig. 3.4.5 Graphical construction of second order motion of a four-bar linkage

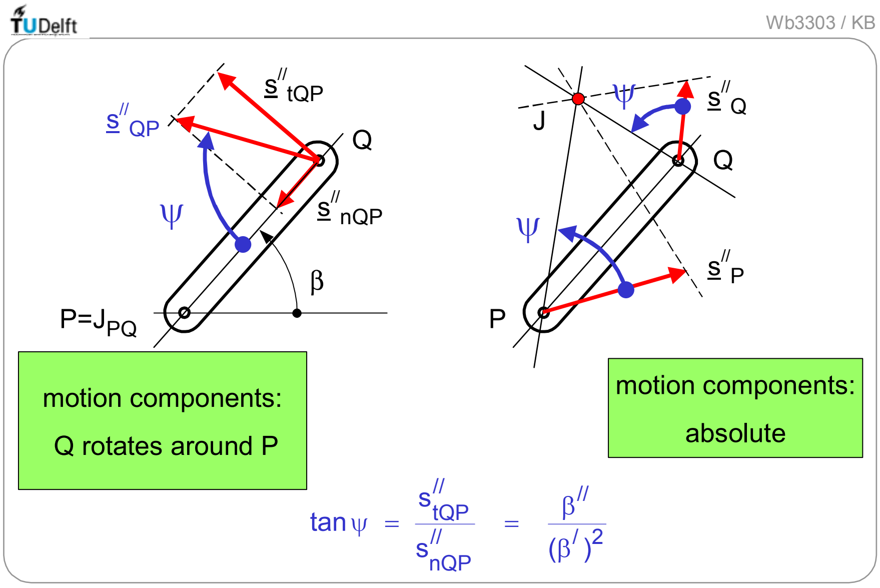


Fig. 3.4.6 Basic procedure for 2nd order pole, using angle ψ

Example 2, a four-bar linkage A_0ABB_0 in the position as drawn in fig. 3.4.5. Crank A_0A is the driving link, so the motion of point A is given according to example 1. The first order motion $\underline{s}^{/}_{B}$ of point B can be found just by drawing a line through A_0 parallel with AB. To find the second order motion $\underline{s}^{//}_{B}$ both links AB and B_0B must be regarded.
In the plane of AB: follow the procedures indicated in figures 3.4.1-3 (find $s^{/}_{BA}$, then $s^{//}_{nBA}$ and draw the first locus for $\underline{s}^{//}_{B}$). In the plane of B_0B the vector $\underline{s}^{/}_{B}$ is itself the first order vector moving around B_0. This vector specifies then the normal component $\underline{s}^{//}_{nBBo}$ and the second locus for $\underline{s}^{//}_{B}$ can be drawn. The intersection of both loci determines the end-point of vector $\underline{s}^{//}_{B}$. Implicitly the angle ψ of the second order vector field has been found, see fig. 3.4.6 left part. For the plane AB for instance it is determined by the components $\underline{s}^{//}_{tBA}$ and $\underline{s}^{//}_{nBA}$, as depicted in fig. 3.4.5 (right part). When the angle ψ will be applied to the known second order vectors of the points A and B (mind the direction), the second order pole J will be found. See for the general idea fig. 3.4.6, right part. With the help of this second order pole other points of the plane AB can be treated easily.

Example 3, a slider-crank mechanism with a gear-and-rack, see fig. 3.4.7. Slider point B (the centre point of the gear) moves at a horizontal line, so the vectors of first and second order are along that line. This line is thus the locus 2 for vector $\underline{s}^{//}_{B}$, which can be constructed almost comparable with the example 2. With $\underline{s}^{/}_{B}$ and $\underline{s}^{//}_{B}$ given, the motion of the wheel can be considered further. It will be clear that the contact point C is the instantaneous centre of rotation, of which the first order vector is known (zero) and the second order vector must be determined. The motion C around B is then to be studied. First the vector $s^{/}_{CB}$ has to be found ($\underline{s}^{/}_{C} = 0$, so $\underline{s}^{/}_{B}$ and $\underline{s}^{/}_{CB}$ are equal but opposite). Next the normal component $\underline{s}^{//}_{nCB}$ can be constructed using the basic construction of fig. 3.4.2 (right case). This determines one locus (locus 2) for $\underline{s}^{//}_{C}$. The other locus (locus 1) is simply the vertical line through C (symmetry reason). With all second order vectors completed, the angle ψ can be reconstructed from the two rotation components $\underline{s}^{//}_{nCB}$ and $\underline{s}^{//}_{tCB}$. This angle can be applied to the two known second order vectors (of points B and C) and the second order pole J will be found.

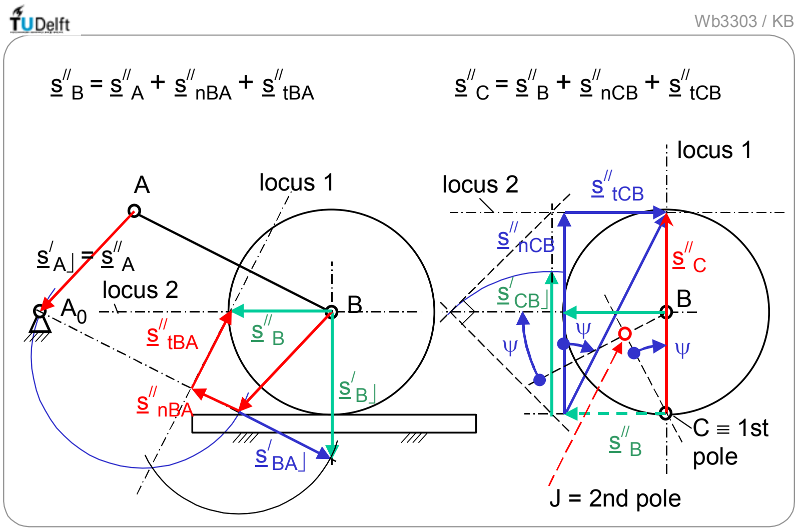


Fig. 3.4.7 *Example: motion components of a mechanism with a gear-and-rack*

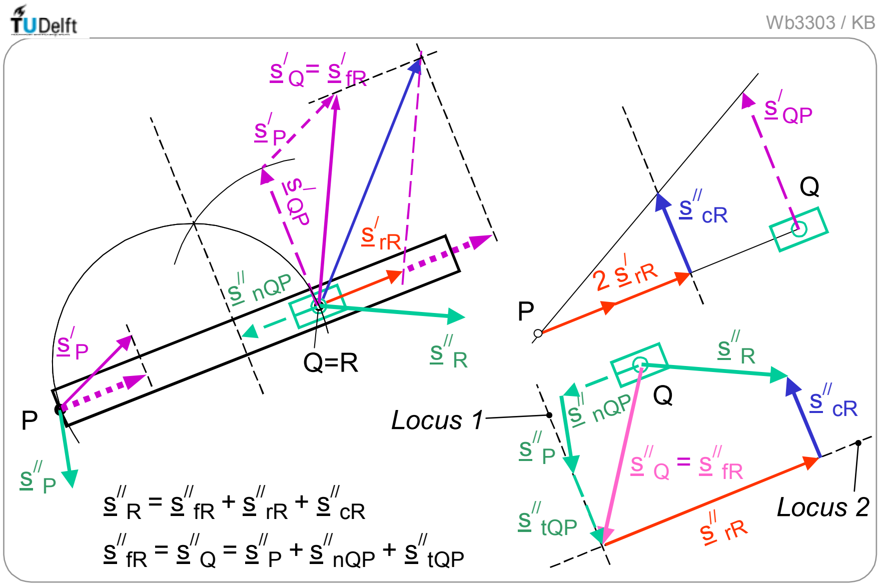


Fig. 3.4.8 Basic procedure for graphical construction of 2nd order relative motion components

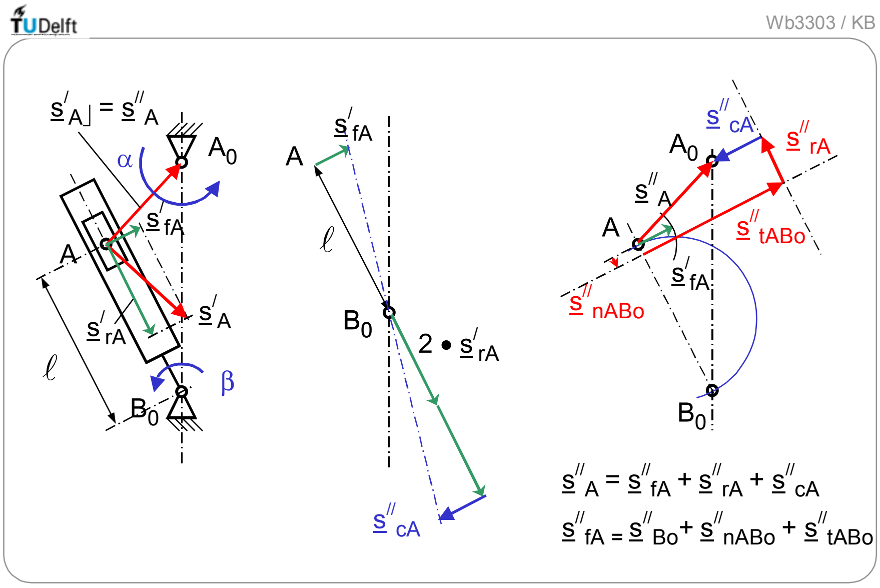


Fig. 3.4.9 Example: inverted slider-crank mechanism, graphical construction of motion components

3.4.2 Relative motion

Continuing with the set-up of relative motion according to fig. 3.2.6, it will be clear that the two points of a slider pair (P and R) can be given any first and second order motion vector. What matters is the decomposition into relative motion and frame motion. The frame motion is for instance necessary to find the rotation component (first and second order derivative of angle β). For better understanding a point Q is defined at the slot, coinciding instantaneously with the slider point R, see fig. 3.4.8. By definition $\underline{s}''_Q$ is the second order frame motion of point R, which has to be found.

The first part of the graphical construction deals with the points P and Q. Because PQ has a fixed distance, the Euler decomposition can be used. After having determined $\underline{s}^{/}_{QP}$ the normal component $\underline{s}^{//}_{nQP}$ can be constructed and the first locus for $\underline{s}^{//}_{Q} = \underline{s}^{//}_{fR}$ can be drawn.
The second part of the graphical construction considers the relative motion decomposition of point R. Among all second order components, $\underline{s}^{//}_{R}$ (the absolute second order vector) is given, while the Coriolis component $\underline{s}^{//}_{cR} = 2\underline{\beta}^{/} \times \underline{\ell}^{/}$ can be constructed in a separated figure (fig. 3.4.8 upper right) using just first order components. The component of relative motion $\underline{s}^{//}_{rR}$ has a known direction and is then the second locus for $\underline{s}^{//}_{fR}$. Figure 3.4.8 (lower part at right) shows the resulting vector sum.

Example, see the inverted slider-crank mechanism of figure 3.4.9. The crank point A has given motion components of first and second order (compare fig. 3.4.4). Point B_0 is a fixed point, so all motion components are zero. Now the graphical construction according fig. 3.4.8 can be carried out. The reader is encouraged to study the result of fig. 3.4.9. It must be remarked that $\ell\beta^{//} = s^{//}_{tABo}$ determines the second order rotation component of the slot rotating around B_0. The angle ψ, as determined by $s^{//}_{tABo}$ and $s^{//}_{nABo}$, is valid both for the slider part and the slot.

3.4.3 Timed motion

For given time derivatives of the input motion quantity, assume crank angle α, the timed motion of all other motion components can be determined afterwards, compare eqs. (1.6) and (1.7).
In the analytical approach all derivatives in the formulas could be taken with respect to time, they would be unchanged except that the primes are replaced by the dots. It should be noticed however that all quantities that are assumed given, must be specified also with respect to time. How this approach differs from the differential motion approach can be explained for instance following the graphical method of mechanism analysis. Consider for instance the example of figure 3.4.5 (four-bar linkage).
Suppose that the angular speed of the crank is given, $\dot{\alpha} = 0.5 \text{rad/s}$, so that $\dot{s}_A = \ell\dot{\alpha} = 0.5\ell$.
All graphical constructions up to order one are the same except for the scale factor 0.5.
Suppose further that the crank has angular acceleration $\ddot{\alpha} = 0.3 \text{rad/s}^2$. The acceleration components of point A become now:

$$\ddot{s}_{tA} = 0.3\ell \text{ and } \ddot{s}_{nA} = (0.5)^2 \ell$$

This shows clearly that the acceleration vector depends on both the values of input velocity and input acceleration. The construction for acceleration is thus *not the same* as a linearly scaled version of the second order geometrical differentiation. Even more problems arise when an arbitrary acceleration vector is drawn for point A. This cannot be done without notice of velocity of point A.
The conclusion is that the scale factors for timed motion must be maintained individually for velocity and acceleration, but they can not be chosen independent of each other.

A comparable problem arises when the graphical construction encounters intersection points outside the paper. A remedy could be to work with shorter vectors, but actually this means that scale factors (smaller velocities or accelerations) are considered. This scale problem is quit comparable with the scale problem in timed motion. A precise interpretation of scale values is required!